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THE SMALLEST HYPERBOLIC 3-MANIFOLDS WITH TOTALLY GEODESIC BOUNDARY

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Dedicated to Professor Akio Hattori on his sixtieth birthday

0. Introduction

A hyperbolic manifold is a riemannian manifold with constant sectional curvature -1. Jørgensen showed that there are only finitely many topological types of the thick part of complete hyperbolic 3-manifolds with bounded volume. Thurston then showed that almost every Dehn surgery on a cusped manifold yields a hyperbolic manifold, and their volumes accumulate at the original cusped manifold from below. These results lead to a global description of the volumes of hyperbolic 3-manifolds which form a well-ordered set of order type ω^ω in several situations [11].

Particular interest has been taken by various authors in the minimum volume. Among others, Meyerhoff [9], Adams [1], and Chinburg and Friedman [3] found the cusped 3-orbifold, the cusped 3-manifold, and the arithmetic 3-orbifold of minimal volume, respectively. In this paper, we will prove

Theorem. *Among compact hyperbolic 3-manifolds with nonempty totally geodesic boundary, each one having the minimum volume admits a polyhedral decomposition by two regular truncated tetrahedra of dihedral angle $\pi/6$.*

The minimum is hence twice the volume of a regular truncated tetrahedron of dihedral angle $\pi/6$. It can be expressed by a definite integral of some elementary functions, and the numerical computation shows that it is $6.452\cdots$. The reader is asked to compare this large value with the other minima. A manifold having the minimum volume is necessarily orientable but not unique, and those manifolds are described by Thurston in [11] and classified by Fujii [5].

We review the polyhedral decomposition in the next section. In §2, we describe the minimum volume, the manifold shown to have the minimum, and its rigidity property in terms of the shape of cut locus. In §3,

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showing a basic inclusion lemma, we obtain a few effective consequences of minimality for finding real targets. Then, using the basic inclusion again, we estimate the volume in §4. In the last section, we show that there is a small stable region for the shape of the cut locus around the manifold in question. This together with the rigid property proves the theorem.

1. Polyhedral decomposition

We begin with recalling some necessary trigonometric rules. The sine and cosine rules for a triangle and the rule for a right angle hexagon, which shows a relation of three nonadjacent edges and another edge, are the basic rules; we call the last rule the hexagon rule. These are described for example in [4], [11]. One more particularly useful rule, which we call the quadrilateral rule, is the relation of three edges of a Lambert quadrilateral:

Quadrilateral Rule. The identity

$$\tanh a = \cosh r \tanh l$$

holds for a Lambert quadrilateral, illustrated in Figure 1.1 with core altitude l, base magnitude r, and side altitude a.

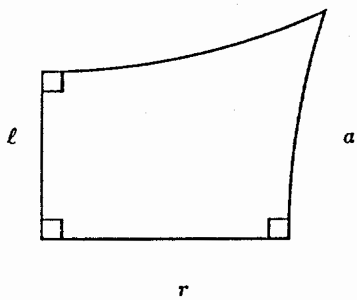


FIGURE 1.1

The following is the notation we use throughout this paper. N denotes a compact hyperbolic 3-manifold with nonempty geodesic boundary ∂N, and D_r a hyperbolic disk of radius r. Note that area

$$D_r = 2\pi(\cosh r - 1).$$

The rest of this section consists of a quick review of the result in [8].

A *return path* is a geodesic segment in N whose end points lie on the boundary components of N (possibly the same boundary component) such that the segment is perpendicular to the boundary at both of its end points. A return path uniquely exists in a free relative homotopy class of paths with respect to the boundary. We call a lift of a return path in the universal cover $\widetilde{N}$ of N a *short cut*; it is the shortest path between the components of $\partial \widetilde{N}$.

There exist only finitely many return paths of bounded length in N. Hence we label them $\lambda_1, \lambda_2, \cdots$ in order of length. Introducing variables $l_j = \text{length}\,\lambda_j$, we get a length sequence $l_1 \le l_2 \le \cdots$ which diverges. The first nonzero shortest distance between terminal points of λ_i and λ_j on ∂N has useful information; we denote it by d_{ij}, which is attained by either an arc or a closed loop. These quantities depend on the manifold N.

The cut locus $\mathbf{C}$ of ∂N in N is the set of points in $\operatorname{int} N$, which admit at least two shortest paths to the boundary. It will be a geometric spine in our setting. $\mathbf{C}$ is stratified by the number of shortest paths which the point admits. The stratification becomes a finite cellular complex structure on $\mathbf{C}$. We hence use the notation $\mathbf{C}$ to mean not only the cut locus, but also the cut locus with this canonical cellular structure.

An appropriate geometric block for N is a truncated polyhedron, which is a compact polyhedron obtained from an ultra-ideal polyhedron by truncating each end (see [5]). We call a face produced by truncation *external* (*internal* otherwise); we also call an edge of an external face *external* (*internal* otherwise). These terminologies are due to the expected location of polyhedra in N. We sometimes call an internal edge a *ridge* for short. Every truncated polyhedron is decomposed by truncated tetrahedra. A truncated tetrahedron is determined by its six dihedral angles up to labelled isometry.

A polyhedral decomposition of N is a finite geometric cellular decomposition of N by truncated polyhedra so that the union of external faces forms the boundary. Every ridge of a polyhedral block is then a return path in N. Only finitely many return paths are involved in the decomposition. The main claim in [8] is

Theorem 1.1 [8]. *The topological dual decomposition of* $\mathbf{C}$ *modulo boundary is homotopic by straightening to a polyhedral decomposition of* N.

By definition the duality is the correspondence between a j-cell of $\mathbf{C}$ and a $(3-j)$-cell of N relative boundary, which transversely intersect each other at an interior point.

Since a return path is unique in a free relative homotopy class of paths, Theorem 1.1 gives a specific decomposition determined by the cut locus. Hence it asserts not only its existence but also its unique resultant. In light of this uniqueness, we call our decomposition *canonical*. The shortest return path λ_1 turns out to be an edge of the canonical polyhedral decomposition.

2. Examples

We describe in this section the manifolds having a polyhedral decomposition by two regular truncated tetrahedra of dihedral angle $\pi/6$. These manifolds will be shown to have the minimum volume.

The Lobachevsky function defined by

$$JI(\theta) = -\int_0^\theta \log|2\sin u|\,du$$

is well known to be convenient for describing volumes of hyperbolic polyhedra. Denote by Δ_θ a regular truncated tetrahedron of dihedral angle θ.

Lemma 2.1.

$$\begin{aligned}\text{volume}\,\Delta_\theta &= 8JI\left(\frac{\pi}{4}\right) - 3\int_0^\theta \operatorname{arccosh}\left(\frac{\cos t}{2\cos t - 1}\right) dt \\ &= \frac{3}{2}\left[JI\left(\frac{\pi}{3}+\phi\right) - JI\left(\frac{\pi}{3}-\phi\right) + JI(\theta+\phi) - JI(\theta-\phi)\right. \\ &\qquad \left. + JI\left(\frac{\pi}{2}-\frac{\theta}{2}-\phi\right) + JI\left(\frac{\pi}{2}+\frac{\theta}{2}-\phi\right) + 2JI\left(\frac{\pi}{2}-\phi\right)\right],\end{aligned}$$

where

$$\tan^2\phi = \frac{\cos^2\theta/2 - \sin^2\pi/3\sin^2\theta}{\cos^2\pi/3\cos^2\theta}.$$

Proof. The above formula for each θ is rather independent. Δ_0 is the ideal regular octahedron, where $\text{volume}\,\Delta_0 = 8JI(\pi/4)$ (see [10], [11]). A one-parameter family $\{\Delta_t : 0 \le t \le \theta\}$ of polyhedra joins Δ_0 with Δ_θ. Then integrating the variation formula of Hodgson [6] for $\text{volume}\,\Delta_t$, we have

$$\text{volume}\,\Delta_\theta - \text{volume}\,\Delta_0 = -\frac{1}{2}\int_0^\theta \text{length(ridges of } \Delta_t)\,dt.$$

An external edge of Δ_t has length $\operatorname{arccosh}(\cos t/(1-\cos t))$ by the cosine rule. Then a ridge of Δ_t has length $\operatorname{arccosh}(\cos t/(2\cos t - 1))$ by the hexagon rule. The first identity follows by substituting it in the integration.

Choose the center of an external face of Δ_θ, and draw the perpendicular path to the internal hexagon on the other side. Regard the path as the core, and cut Δ_θ into six congruent doubly truncated orthoschemes of dihedral angles $\pi/3$, θ, and $\theta/2$. Each edge of the dihedral angle $\theta/2$ joins the faces produced by the truncation. The second identity now follows from the volume formula of such a truncated orthoscheme by Kellerhals [7]. q.e.d.

Take two copies of $\Delta_{\pi/6}$, and glue them along internal faces so that the resultant is a nonsingular hyperbolic manifold with geodesic boundary. There actually are several ways to get the desired resultants; denote any one of them by N_0. N_0 is hence a notation for a manifold in the class specified here. It is easy to see that N_0 is necessarily orientable. It was shown in [5] that there are exactly eight orientable isometry types in this class.

There is some numerical information of N_0 in common. The shortest return path λ_1 of N_0 is a ridge of the tetrahedra $\Delta_{\pi/6}$ in the decomposition. Applying the cosine rule to an external face and then the hexagon rule to an internal face, we get $\cosh l_1 = (3+\sqrt{3})/4 = 1.1830\cdots$. Obviously $\text{volume}\, N_0 = 2\,\text{volume}\, \Delta_{\pi/6}$. Numerical computation using the identities in Lemma 2.1 by 'Mathematica' shows that it is $6.452\cdots$.

The cut locus $\mathbf{C}$ of N_0 is very simple; it contains only one 2-cell. Hence, by duality there is only one return path involved in the canonical polyhedral decomposition. Since this simplicity is frequently used in the sequel, let us call a cut locus *simple* if it has only one 2-cell. This simple structure is related to the following rigid property of N_0.

Lemma 2.2. *Denote the Euler characteristic of ∂N by $\chi(\partial N)$. If $\mathbf{C}$ is simple and $\chi(\partial N) = -2$, then the canonical polyhedral decomposition of N consists of either two regular truncated tetrahedra with dihedral angle $\pi/6$ and in particular $N = N_0$, or one regular truncated octahedron with dihedral angle $\pi/6$.*

Proof. Since $\mathbf{C}$ is simple, there is only one return path involved in the canonical polyhedral decomposition. Hence the induced decomposition on ∂N has exactly two vertices which correspond to the end points of this path. Let e and f be the number of edges and faces on ∂N respectively. Then

$$-2 = \chi(\partial N) = 2 - e + f \le 2 - 3f/2 + f,$$

which implies $f \le 8$. Since each polyhedron has at least four external faces, the number of polyhedra involved in the decomposition is at most two.

If there are two polyhedra in the decomposition, then they must be truncated tetrahedra by a standard Euler characteristic argument.

Moreover every ridge has the same length because they are identified in N, and hence the polyhedra must be regular. Since there are twelve ridges gathered at the shortest return path, the dihedral angle of each ridge turns out to be $2\pi/12 = \pi/6$.

Suppose that there is only one polyhedron in the decomposition, and denote it by P. It is a truncated polyhedron with at most eight external faces. P has six internal faces. To see this, by collapsing each external face of P to a point, we get a ball P^*. The combinatorial structure of P^* is realized by a piecewise linear cell. The number of its faces is equal to $\#\{\text{edges}\} - \#\{\text{vertices}\} + 2$ by Euler's identity. By the definition of P^*, we have $\#\{\text{vertices}\} = f$ and $2\#\{\text{edges}\} = \#\{\text{external edges of } P\} = 2e$. Since $e = f + 4$, we hence get $\#\{\text{faces}\} = 6$.

Also, the internal faces of P consist of pairs of isometric polygons, since N is obtained by gluing internal faces of P. Thus the five polyhedra listed in Figure 2.1 are the only possible combinatorial types for P^*. Another polyhedron either has more than eight vertices or does not have faces in pairs.

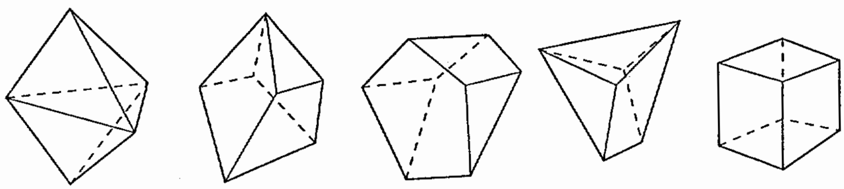


FIGURE 2.1

Let us realize P^* concretely by a piecewise linear cell inscribed in the unit sphere. Since P is the only polyhedron to form the canonical polyhedral decomposition of N, the cut locus $\mathbf{C}$ has only one vertex v by duality. Put v at the center of the Poincaré disk. There are f shortest rays from v to the boundary of the universal cover of N. Extend them to the sphere at ∞ and realize P^* by the linear convex hull of their end points.

The length of an internal edge of P is the distance of the components of $\partial\widetilde{N}$ which the associated rays go through. Since each ray to the component of $\partial\widetilde{N}$ has the same length, the distance is a monotone function in terms of the angle which the associated rays bound. Recall that each internal edge of P has the same length since they all are identified in N. Hence the corresponding angles all must be the same, so that P^* inscribed in the unit sphere has equilateral edges. This restriction rules out all the possible combinatorial cells in Figure 2.1 except the last one.

Therefore P^* is necessarily an equilateral hexahedron inscribed in the unit sphere. Since each face is an equilateral quadrilateral inscribed in the circle, it is a square, P^* is a cube, and P is a regular truncated hexahedron. In particular, all the ridges have the same dihedral angle. There are twelve ridges gathered at the shortest return path, and hence the dihedral angle of each ridge turns out to be $2\pi/12 = \pi/6$. q.e.d.

3. Basic inclusion

The core of this section is a basic inclusion established in Lemma 3.2. We will use the basic inclusion to prove a number of results culminating in

Proposition 3.1. *If the volume of* N *is minimal, then* $\chi(\partial N) = -2$ *and* $\cosh l_1 \geq (3+\sqrt{3})/4$.

Reflecting a Lambert quadrilateral along the top edge, we get a symmetric pentagon. Let λ be the left edge of the pentagon whose shape is determined by $l = \operatorname{length}\lambda$ and the bottom magnitude. Take a right angle hexagon whose three nonadjacent edges all have length l. Then we define a preferred bottom magnitude R to be half the length of another edge of the hexagon (see Figure 3.1). A preferred bottom magnitude produces a very special symmetric pentagon; in particular, it has an angle of $2\pi/3$. By the hexagon rule, R is a function of l and hence $x = \cosh l$. More precisely,

$$\cosh R = \sqrt{\frac{2\cosh l - 1}{2(\cosh l - 1)}} = \sqrt{\frac{2x-1}{2(x-1)}}.$$

The side altitude A of the based Lambert quadrilateral is also a function of l or x and satisfies

$$\cosh 2A = \frac{4x+1}{3}.$$

Consider the hyperbolic solid which is the solid of revolution of the above very special symmetric pentagon about λ. R is a radius of the top and also bottom disks (see Figure 3.1, next page). We call it an *English muffin* or simply *muffin* because of the shape, and denote it by M_l. The variable in the notation is sufficient since R is a function of l.

To see the relation of a muffin with N, recall that λ_1 is the shortest return path of N, and l_1 is its length. The related constants $x_1 = \cosh l_1$, $R_1 = R(x_1)$, and $A_1 = A(x_1)$ depend on N. Here we have the basic inclusion lemma.

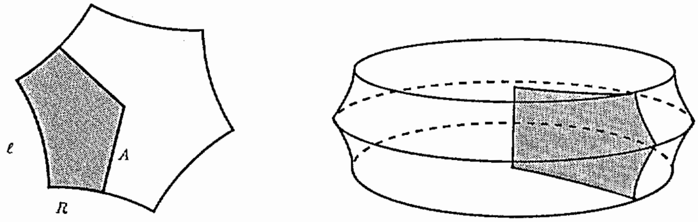


FIGURE 3.1

Lemma 3.2. *M_{l_1} can be packed in N so that the core of M_{l_1} is matched up with the shortest return path λ_1. In particular,*

$$\text{volume}\, N \geq \text{volume}\, M_{l_1},$$

and the top and bottom disks at M_{l_1} are packed in ∂N.

Proof. Since λ_1 is simple, a muffin-like neighborhood of λ_1 is embedded. Hence the question is to estimate how much it can be thickened. In the universal covering space, we have lifts of λ_1, which are mutually disjoint short cuts.

Recall the fact that three disjoint geodesic surfaces in $\mathbf{H}^3$ uniquely determine a right angle hexagon spanned by the short cuts between them. Or two short cuts having terminal points in a common component of $\partial\widetilde{N}$ span a right angle hexagon in $\widetilde{N}$.

d_{11} is the first nonzero shortest distance between terminal points of λ_1 and λ_1, and is the shortest distance between terminal points of lifts of λ_1 in $\widetilde{N}$. Choose lifts $\tilde{\lambda}_1$ and $\tilde{\lambda}'_1$ of λ_1 so that they touch a common component of $\partial\widetilde{N}$ and attain the distance d_{11} there. These two lifts determine a right angle hexagon. The last short cut involved has length $l \geq l_1$ because l_1 is the shortest length of the short cuts. We thus have

$$\begin{aligned} \cosh d_{11} &= \frac{\cosh^2 l_1 + \cosh l}{\sinh^2 l_1} \\ &\geq \frac{\cosh^2 l_1 + \cosh l_1}{\sinh^2 l_1} = \cosh 2R_1, \end{aligned}$$

which implies $d_{11} \geq 2R_1$, and we can thicken a thin muffin up to one with base radius R_1 at least on the boundary.

To see that the thickening yields a packing even in $\text{int}\, N$, assume the contrary. Then there are lifts $\tilde{\lambda}_1$, $\tilde{\lambda}'_1$ and their thickened muffins M, M' in $\widetilde{N}$ with base radius R_1 so that $\text{int}\, M$ and $\text{int}\, M'$ have a common point v. Since the thickening defines a packing on $\partial\widetilde{N}$, the terminal points of $\tilde{\lambda}_1$ and $\tilde{\lambda}'_1$ are on mutually distinct components of $\partial\widetilde{N}$.

Draw perpendicular paths from v to the top and bottom disks of M, and denote their end points by p and q so that length $pv \geq$ length qv. qv necessarily has less length than A_1. The same procedure for M' supplies the points p' and q' on its top and bottom disks so that length $p'v \geq$ length $q'v$. Then consider a pentagon spanned by $\tilde{\lambda}_1$ and a sector pvq. The pentagon has right angle vertices, except v, and is placed in the basic pentagon of M where v is in its interior. In particular, $\angle pvq > 2\pi/3$ by the definition of R_1. Similarly, $\angle p'vq' > 2\pi/3$.

Next let $\tilde{\lambda}_{qq'}$ be the short cut which connects a component of $\partial\widetilde{N}$ containing q with another component containing q'. $\tilde{\lambda}_{qq'}$ and v span a geodesic plane containing both q and q'. Thus $\tilde{\lambda}_{qq'}$ and qvq' bound a pentagon with four right angle vertices. $\tilde{\lambda}_{qq'}$ has length $\geq l_1$. Both qv and $q'v$ have length less than A_1. Put this pentagon on the basic pentagon of M so that v is at the vertex with an angle of $2\pi/3$ and q is on an edge. If q' is contained in the basic pentagon, then $\tilde{\lambda}_{qq'}$ must have length less than l_1, which is a contradiction. Hence q' lies outside the basic pentagon and in particular the angle $\angle qvq'$ is larger than $2\pi/3$.

We then assume without loss of generality that length $pv \leq$ length $p'v$, and consider the short cut $\tilde{\lambda}_{pq'}$ which connects a component of $\partial\widetilde{N}$ having p with another component having q'. $\tilde{\lambda}_{pq'}$ and v span a pentagon with four right angle vertices. Fold it on the pentagon spanned by $\tilde{\lambda}'_1$ and $p'vq'$ along the folder vq'. Since length $\tilde{\lambda}_{pq'} \geq$ length $\tilde{\lambda}'_{-1} = l_1$, p is forced to be outside this pentagon and hence $\angle pvq' > \langle p'vq' > 2\pi/3$.

We thus get three adjacent angles $\angle pvq$, $\angle qvq'$, and $\angle q'vp$ at v which are all $> 2\pi/3$. This is impossible. q.e.d.

Lemma 3.3.

$$\begin{aligned}\text{volume}\, M_l &= 2\pi\left(A\cosh R - \frac{l}{2}\right)\\ &= \pi\left(\sqrt{\frac{2x-1}{2(x-1)}}\operatorname{arccosh}\frac{4x+1}{3} - \operatorname{arccosh} x\right),\end{aligned}$$

and it is monotone decreasing in terms of x.

Proof. The half muffin cut by the orthogonal bisector to the core λ is a solid of revolution of a Lambert quadrilateral of core altitude $l/2$, base magnitude R, and side altitude A, and its volume is $\pi(A\cosh R - l/2)$ (see p. 213 of [4]). Then the claim is an exercise in calculus. q.e.d.

We have two corollaries of the basic inclusion, which prove Proposition 3.1.

Corollary 3.4. *If* $\chi(\partial N) < -2$, *then* volume $N >$ volume N_0. *In particular, if* volume N *is minimal, then* $\chi(\partial N) = -2$.

Proof. By the numerical computation of volume M_l at $x = 1.14$, we have that volume $N >$ volume $M_{l_1} > 6.5$ if $x_1 \leq 1.14$ since volume M_l is monotone decreasing. Hence consider the other case $x_1 \geq 1.14$ or more roughly $l_1 = \operatorname{arccosh} x_1 > 0.52$. The collar of the boundary with depth $l_1/2$ is embedded in N by the definition of l_1. We thus have volume $N >$ area $\partial N \cdot l_1/2 = \pi|\chi(\partial N)| \cdot l_1$. Since $\chi(\partial N) = 2\chi(N)$ is even, the assumption implies that $\chi(\partial N) \leq -4$, and $\pi|\chi(\partial N)| \cdot l_1 > 6.5$. In either case, volume $N > 6.5 >$ volume N_0. The last claim follows since the maximal Euler characteristic is -2. q.e.d.

We recall here a result in [2]. Start with a packing of $\mathbf{H}^2$ by disks of radius R. The set of points which admit at least two shortest paths to the disks of the packing defines a polygonal decomposition of $\mathbf{H}^2$. The *local density* of the packing is a density for each member of the packing defined by the ratio of area D_R by the area of the polygon bounding the member in question. Consider three disks of radius R on $\mathbf{H}^2$ touching each other. Their centers span an equilateral triangle Δ of edge length $2R$ and angle α. By the cosine rule, α is related to x and hence l by

$$\cos\alpha = \frac{\cosh 2R}{\cosh 2R + 1} = \frac{x}{2x - 1}.$$

Then Böröczky showed in [2] that the local density of any packing by disks of radius R is bounded by

$$\frac{\operatorname{area}(\Delta \cap \text{disks})}{\operatorname{area}\Delta},$$

which is a function of x or l.

Corollary 3.5. *If* $\chi(\partial N) = -2$, *then* $x_1 \geq (3 + \sqrt{3})/4$.

Proof. We had a packing on ∂N by two disks of radius R_1 by the basic inclusion. Then Böröczky's bound works also for this case, and we get an estimate of $\alpha_1 = \alpha(x_1)$ by area comparison,

$$2\operatorname{area} D_{R_1} \leq 2\pi|\chi(\partial N)| \cdot (\text{Böröczky's bound}),$$

which implies $\alpha_1 \geq \pi/6$, and hence $x_1 \geq (3 + \sqrt{3})/4$. q.e.d.

4. Volume estimate

In this section, by estimating the contribution of the complement of an embedded muffin, we improve the volume estimate in the basic inclusion.

In particular, we have

Proposition 4.1. *If* $\chi(\partial N) = -2$ *and* $\cosh l_1 \geq 1.186$, *then* volume N > volume N_0.

Recall that l_2 is the length of the second shortest return path of N. l_2 is not a function of l_1, however we will have a bound. Also recall that d_{ij} is the first nonzero shortest distance of the terminal points of λ_i and λ_j. We have shown previously that $2R_1 \leq d_{11}$.

Lemma 4.2.

$$\cosh l_2 \geq \frac{2}{\cosh^2 d_{12} \tanh^2 l_1 - 1} + 1,$$

$$\cosh l_2 \geq \sqrt{\frac{\cosh l_1 + 1}{\cosh d_{22} - 1}} + 1.$$

Proof. Choose lifts $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of the return paths so that they touch a common component of $\partial \widetilde{N}$ and attain the distance d_{12} there. Since a hexagon determined by $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ has the last short cut of length $l \geq l_1$, by the hexagon rule we have an inequality

$$\cosh d_{12} \geq \frac{\cosh l_1 \cosh l_2 + \cosh l_1}{\sinh l_1 \sinh l_2}.$$

The first estimate in the statement is identical to this. The second is a consequence of the same argument for a hexagon determined by $\tilde{\lambda}_2$ and $\tilde{\lambda}_2'$, which touch a common component of $\partial \widetilde{N}$ and attain the distance d_{22} there. q.e.d.

The top and bottom of a packed muffin by the basic inclusion are disks packed in ∂N; denote these two by U and U'. R_1 is the common radius of U and U' and obviously bounds the maximum radius of two disks packed in ∂N from below. We introduce here a function R' of l or x defined by

$$\cosh R' = 3 - \cosh R.$$

$R'(x_1) = R_1'$ gives a very optimistic upper bound of the radius of two disks packed in $\partial N - \operatorname{int}(U \cup U')$. In fact, we have

Lemma 4.3. *If* $\chi(\partial N) = -2$, *then the maximum radius of two disks packed in* $\partial N - \operatorname{int}(U \cup U')$ *is less than* R_1'. *Also either* $R_1 + R_1' \geq d_{12}$ *or* $2R_1' \geq d_{22}$.

Proof. Suppose two disks of radius R_1' are packed in $\partial N - \operatorname{int}(U \cup U')$. Then, since there must be complementary room, we have

$$2 \operatorname{area} D_{R_1} + 2 \operatorname{area} D_{R_1'} < 2\pi |\chi(\partial N)|,$$

which contradicts the definition of R_1'.

The maximum radius of two disks packed in $\partial N - \text{int}(U \cup U')$ having their center at the terminal points of l_2 is obviously less than such radius with no restriction on the location of centers. Hence it is also less than R'_1. The paired maximal disks in $\partial N - \text{int}(U \cup U')$ with the preferred centers must touch either U, U', or themselves. If one of them touches U or U', then $R_1 + R'_1 \geq d_{12}$. Otherwise, either they touch each other or one of them touches itself, and $2R'_1 \geq d_{22}$ in either case. q.e.d.

Thus define two functions of l or x by

$$\cosh E = \frac{2}{\cosh^2(R+R') \cdot \tanh^2 l - 1} + 1,$$

$$\cosh F = \sqrt{\frac{\cosh l + 1}{\cosh 2R' - 1} + 1}.$$

If we let $E(x_1) = E_1$ and $F(x_1) = F_1$, then $l_2 \geq \min\{E_1, F_1\}$ by Lemmas 4.2 and 4.3.

Proof of Proposition 4.1. By estimating the contribution of the collar of $\partial N - \text{int}(U \cup U')$, we will find a better lower bound of volume N.

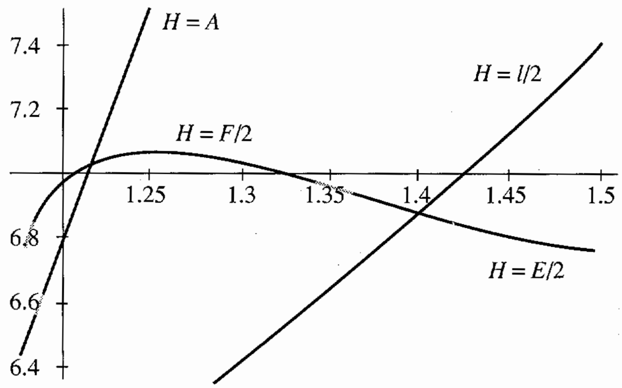
$A(x_1) = A_1$ is the side altitude of half of the muffin M_{l_1}. The depth of the collar of $\partial N - \text{int}(U \cup U')$ is equal to $\min\{l_2/2, A_1\}$, but unfortunately it is not a function of x_1. Since $l_2 \geq \min\{E_1, F_1\}$, we have $\min\{l_2/2, A_1\} \geq \min\{\min\{E_1/2, F_1/2\}, A_1\}$, where the right-hand side is now a function of l_1 or x_1 and bounds the depth of the collar from below. In the product metric space, the volume of the collar is obviously the product of the area of the base and the depth of the collar. Replacing 'depth' by $(2\,\text{depth} + \sinh(2\,\text{depth}))/4$, we get a hyperbolic volume of the collar (see p. 211 of [4]). Hence, letting $H = \max\{\min\{E/2, F/2, A\}, l/2\}$ and $H(x_1) = H_1$, we improve the volume estimate by

$$\text{volume}\, N \geq \text{volume}\, M_{l_1} + (4\pi - 2\,\text{area}\, D_{R_1}) \cdot \frac{2H_1 + \sinh 2H_1}{4}.$$

The graph of the function

$$\text{volume}\, M_l + (4\pi - 2\,\text{area}\, D_R)(2H + \sinh 2H)/4$$

in terms of x is described in Graph 4.1. The numerical computation shows that the value at $x = 1.186$ is $6.47\cdots$ and attains the minimum on $x \geq 1.186$. In particular, volume $N >$ volume N_0 in $x_1 \geq 1.186$. q.e.d.



GRAPH 4.1

5. Simple cut locus and proof of theorem

In this section, we fill up the gap between $(3+\sqrt{3})/4$ and 1.186 by showing the proposition below, and prove the Theorem.

Proposition 5.1. *If* $\chi(\partial N) = -2$ *and* $(3+\sqrt{3})/4 \le x_1 \le 1.186$, *then the cut locus of* N *is simple.*

Consider again an equilateral triangle Δ of edge length $2R$, and let J be the distance between the center and a vertex. It is a function of l and hence x. Obviously $J > R$ and satisfies

$$\cosh J = \sqrt{\frac{3\cosh l - 1}{3(\cosh l - 1)}} = \sqrt{\frac{3x-1}{3(x-1)}}.$$

Recall that U and U' are two disks in ∂N centered at the terminal points of λ_1. By thickening these to disks V and V' with the same centers so that the radius is $J_1 = J(x_1)$, we get an overlapped configuration of two disks on ∂N. If $N = N_0$, then these entirely cover ∂N_0.

A component of $V \cap V'$ is generically a round bigon since U and U' are packed in ∂N. By a round polygon, we mean a polygon bounded by parts of circles. Three bigons might touch at their vertices. This occurs typically on ∂N_0. In general, $V \cap V'$ is a union of bigons with some common vertices. The centers of V and V' have distance at least $2R$. Hence the area of the bigon is bounded by that of the maximally overlapped configuration in Figure 5.1 (next page), though it may not be realized by some N. The area B of this bigon is then a function of l or x defined by

$$B = \frac{2\beta}{2\pi} \cdot \operatorname{area} D_J - (2\pi - 2\beta - 2\alpha).$$

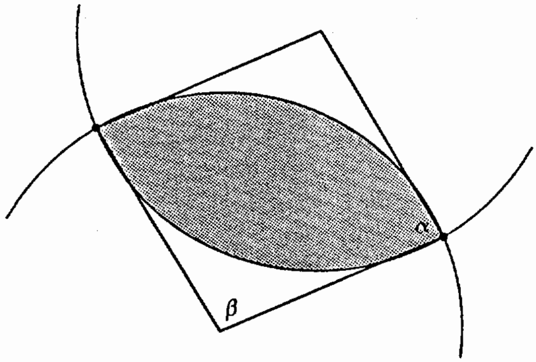


FIGURE 5.1

Each bigon contains a core geodesic connecting the two vertices. To the core of each bigon in $V \cap V'$, we assign the orthogonal bisector on $\partial N - \text{int}(U \cup U')$ which bridges the boundary of $U \cup U'$. Then slit $\partial N - \text{int}(U \cup U')$ along these bisectors to get a jigsaw puzzle on $\partial N - \text{int}(U \cup U')$, and each slitted region is a piece of the puzzle.

A piece of the puzzle is also a round polygon, where the slits are geodesics. There roughly is a one-to-one correspondence between pieces of the puzzle and round polygonal components of $\partial N - V \cup V'$ by inclusion. These regions are linked as in Figure 5.2, and we call the included round polygon a *hole*. A hole is located in a piece of the puzzle as a kernal. The puzzle on ∂N_0 is very special, and it consists of twelve round triangular pieces without holes.

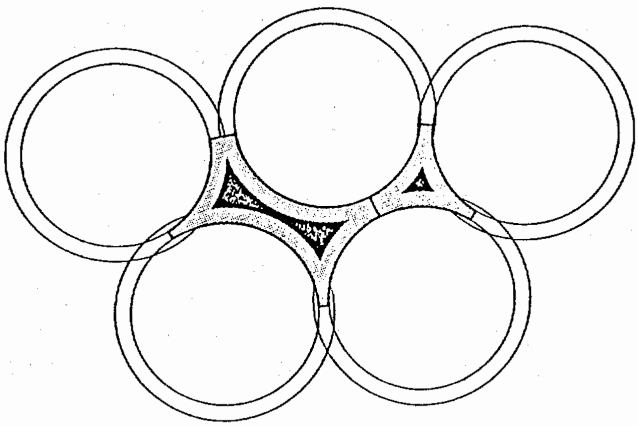


FIGURE 5.2

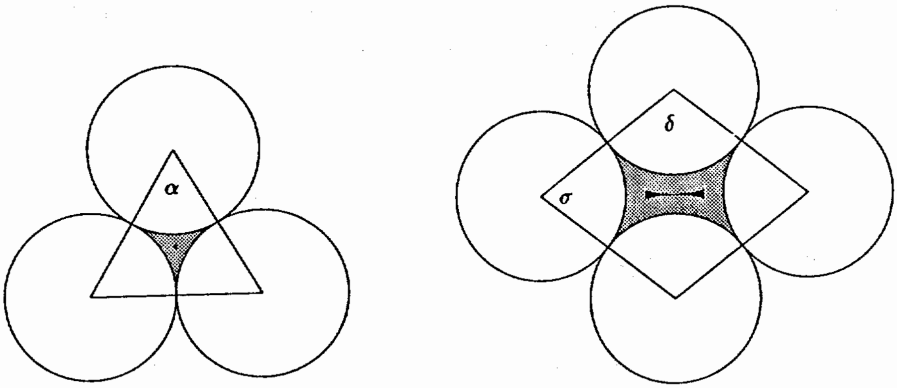


FIGURE 5.3

It is quite conceivable that the holes are all triangular if x_1 is close enough to $(3+\sqrt{3})/4$. In fact, we have

Lemma 5.2. *If* $\chi(\partial N) = -2$ *and* $\chi_1 \leq 1.186$, *then* $\partial N - V \cup V'$ *consists of triangular holes.*

Proof. Let b be the number of bigons in $V \cap V'$, and decompose $V \cup V'$ by edges, which joins vertices in each bigon. Let c be the number of vertices at which three bigons meet. Then $\partial(V \cup V') = \partial$ (holes) has $2b-3c$ edges, and $\chi(V \cup V') = ((2b-3c)+c)-(b+(2b-3c))+2 = -b+c+2$. Also since each hole has at least three edges and its Euler characteristic is at most 1, χ (holes) $\leq (2b-3c)/3$. Thus $\chi(\partial N) = \chi(V \cup V') + \chi$ (holes) $\leq 2-b/3$, and we get $b \leq 12$. On the other hand, we have an obvious bound:

$$\begin{aligned}\text{area(holes)} &= \text{area(bigons)} - 2\,\text{area}\, D_J + \text{area}\,\partial N \\ &\leq b \cdot B - 4\pi \cosh J + 8\pi .\end{aligned}$$

The numerical computation shows that the last term is negative if $b \leq 10$ and $x_1 \leq 1.186$, which is absurd. Thus $b = 11$ or 12. When $b = 12$, χ(holes) $= 8 - c$ and the holes have $24 - 3c$ edges in total. Hence they consist of $8 - c$ triangles. When $b = 11$, χ(holes) $= 7 - c$ and the holes have $22 - 3c$ edges in total. Hence they consist of $6 - c$ triangles and one quadrilateral.

To rule out the latter case, we consider an abstractly smallest possible piece of a puzzle containing a triangular or quadrilateral hole. It is obvious that a piece containing a triangular hole has more area than the area of a piece without a hole, which is illustrated by the shaded region on the left side of Figure 5.3. It is not quite obvious but easy to see that a piece containing a quadrilateral hole has more area than the area of the shaded region of the right configuration in Figure 5.3, where the top and bottom circles of radius J touch each other.

The areas of the configuration in the figure, which we denote by T and Q, are functions of l or x defined by

$$T = (\pi - 3\alpha) - \frac{3\alpha}{2\pi} \cdot \operatorname{area} D_R,$$

$$Q = (2\pi - 2\delta - 2\sigma) - \frac{(2\delta + 2\sigma)}{2\pi} \cdot \operatorname{area} D_R.$$

Then we should have an inequality

$$2 \operatorname{area} D_R + 6T + Q \leq \operatorname{area}(\partial N) = 4\pi$$

by area comparison if the case in question occurred. But the left-hand side is monotone decreasing in terms of x, and the numerical computation shows that the value at $x = 1.186$ is $12.59\cdots > 4\pi$. Hence the case cannot be realized. q.e.d.

The last useful and very neat function, S, for our purpose is

$$\cosh S = \sqrt{\frac{\cosh l}{\cosh l - 1}} = \sqrt{\frac{x}{x-1}},$$

which is identified with the diagonal length of a symmetric Lambert quadrilateral illustrated in Figure 5.4.

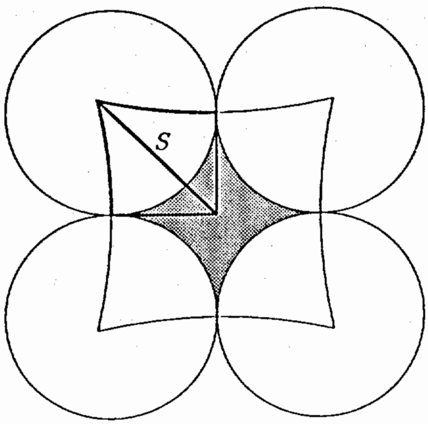


FIGURE 5.4

To see a very useful property of $S(x_1) = S_1$, we let d_f be $\max(x, y)$ where $x \in \partial N$ and y is a terminal point of λ_1.

Lemma 5.3. *If $d_f \leq S_1$, then the shortest distance of any point in N to the boundary is $\leq l_2/2$. In particular, the cut locus is simple.*

Proof. Recall the estimate of d_{12} in Lemma 4.2, it is simplified to

$$\cosh^2 d_{12} \geq \coth^2 l_1 \left(\frac{2}{\cosh l_2 - 1} + 1\right).$$

By the assumption, we have

$$\frac{x_1}{x_1 - 1} \geq \cosh^2 d_f \geq \cosh^2 d_{12}.$$

Combining these, we get an estimate

$$\cosh l_2 \geq 2x_1 + 1,$$

or equivalently,

$$\tanh^2 \frac{l_2}{2} \geq \frac{x_1}{x_1 + 1}.$$

On the other hand, the edge altitude a of a Lambert quadrilateral with base magnitude d_f $(\leq S_1)$ and core altitude $l_1/2$ is related to x_1 by

$$\begin{aligned}\tanh^2 a &= \cosh^2 d_f \tanh^2 \frac{l_1}{2} \\ &\leq \frac{x_1}{x_1 - 1} \cdot \frac{x_1 - 1}{x_1 + 1} = \frac{x_1}{x_1 + 1},\end{aligned}$$

which implies $a \leq l_2/2$. This means that the 2-cell involved in the cut locus is only one on the bisector of l_1.

Proof of Proposition 5.1. Since $x_1 \leq 1.186$, every hole must be triangular by Lemma 5.2. The point which attains d_f is hence contained in some triangular hole. The distance d_{tri} between the vertex and the center of the equilateral triangle of edge length $2J_1$ bounds d_f from above. By the sine rule, we have

$$\frac{\sinh d_{\text{tri}}}{\sin \pi/3} = \frac{\sinh J_1}{\sin \pi/2},$$

which implies

$$\cosh d_f \leq \cosh d_{\text{tri}} = \sqrt{\frac{9x_1 - 1}{9(x_1 - 1)}} < \cosh S_1.$$

Then by Lemma 5.3, the cut locus is simple.

Proof of Theorem. Suppose N has the minimum volume. Then by Propositions 3.1, 4.1, and 5.1, $\chi(\partial N) = -2$ and the cut locus is simple. By the rigidity established in Lemma 2.2, either $N = N_0$ or N has a canonical polyhedral decomposition by one regular octahedron of dihedral angle $\pi/6$. The latter case is ruled out since then the shortest return path

λ_1 is realized by a ridge with $\cosh l_1 = (1+\sqrt{3})/2 = 1.366\cdots$, and the volume is greater than volume N_0 by Proposition 4.1. q.e.d.

References

[1] C. Adams, *The noncompact hyperbolic* 3-*manifold of minimal volume*, Proc. Amer. Math. Soc. **100** (1987) 601–606.

[2] K. Böröczky, *Packing of spheres in spaces of constant curvature*, Acta Math. Acad. Sci. Hungar. **32** (1978) 243–261.

[3] T. Chinburg & E. Friedman, *The smallest arithmetic hyperbolic three-orbifold*, Invent. Math. **86** (1986) 507–527.

[4] W. Fenchel, *Elementary geometry in hyperbolic spaces*, de Gruyter Studies in Math., No. 11, Walter de Gruyter, Berlin, 1989.

[5] M. Fujii, *Hyperbolic* 3-*manifolds with totally geodesic boundary which are decomposed into hyperbolic truncated tetrahedra*, Tokyo J. Math. **13** (1990) 353–373.

[6] C. Hodgson, *Degeneration and regeneration of geometric structures on three-manifolds*, Thesis, Princeton Univ., 1986.

[7] R. Kellerhals, *On the volume of hyperbolic polyhedra*, Math. Ann. **285** (1989) 541–569.

[8] S. Kojima, *Polyhedral decomposition of hyperbolic* 3-*manifolds with totally geodesic boundary*, to appear in Advanced Studies in Pure Math., Vol. 20.

[9] R. Meyerhoff, *The cusped hyperbolic* 3-*orbifold of minimum volume*, Bull. Amer. Math. Soc. (N.S.) **13** (1985) 154–156.

[10] J. Milnor, *Hyperbolic geometry: The first* 150 *years*, Bull. Amer. Math. Soc. (N.S.) **6** (1982) 9–24.

[11] W. P. Thurston, *The geometry and topology of* 3-*manifolds*, Lecture Notes, Princeton Univ., 1977/78.

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